EN_49: — The discrete-continuous link through equivalence classes

UNIHOLOG

This document reconstructs the development of EN_49 on the link between the discrete and the continuous, formulated without falsely identifying N with R.

The central point is not that the set of natural numbers and the set of real numbers are equivalent as structures. They are not. The central point is more precise: a discrete construction indexed by N and a continuous construction defined over R can determine exactly the same mathematical object of R.

That common object acts as the link.

In EN_49, this link is expressed through families or equivalence classes of the type U_e , U_π , U_φ , $U_{\tilde{n}}$, U_g and, in general form, U_{f_i} .

All formulas in this document are written using only `<math xmlns="http://www.w3.org/1998/Math/MathML" display="inline"><mrow><mo>.</mo><mo>.</mo><mo>.</mo></mrow></math>` for inline expressions and `<math xmlns="http://www.w3.org/1998/Math/MathML" display="inline"><mrow><mo>.</mo><mo>.</mo><mo>.</mo></mrow></math>` for blocks.

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1. Starting point

Let $\alpha \in R$ be a real constant.

The same constant may appear through expressions of different nature.

For example, it may appear as the limit of a sequence:

$$\lim_{n \to \infty} a_n = \alpha,$$

where the index n belongs to N.

It may also appear through a continuous process:

$$\lim_{dx \to 0} Q(F, x, dx) = \alpha,$$

where x and dx belong to the framework of variation of R.

The first construction uses a discrete structure:

$$n = 1, 2, 3, \ldots$$

The second uses a continuous structure:

$$dx \to 0 .$$

However, if both constructions determine the same value α, then both belong to the same class with respect to their value.

The link is not an identity between domains.

The link is the common value.

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2. Fundamental distinction between finite expression, limit process and value

Before defining the classes U_α, it is useful to separate three levels.

2.1 Finite expression depending on a parameter

An expression such as

$$\left(1+\frac{1}{n}\right)^n$$

depends on n.

For a finite value of n, it is generally not equal to its limit.

Therefore,

$$\left(1+\frac{1}{n}\right)^n \neq e$$

for every finite n.

2.2 Limit process

The complete process

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

does determine exactly e.

The finite expression does not belong to U_e as an exact expression, but the complete limit process does belong to U_e.

2.3 Determined value

The value determined by the process is the mathematical object:

$$e\,.$$

Thus, EN_49 distinguishes:

1. finite approximations;
2. limit processes;
3. exact values determined by those processes.

This distinction prevents confusing a sequence of approximations with the constant itself.

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3. Definition of U_α

Let $\alpha \in R$.

We define U_α as the set of expressions, definitions, processes or mathematical constructions that determine exactly the value α.

Formally:

$U_\alpha = \{A\colon \text{valor}\,(A) = \alpha\}\,.$

Here $\text{valor}\,(A)$ must be understood as the mathematical value determined by A, whenever that value is well defined.

If A is an exact expression, then $\text{valor}\,(A)$ is its direct value.

If A is a limit process, then $\text{valor}\,(A)$ is the limit, provided that it exists.

For example:

$A = \sum_{k=0}^{\infty} \frac{1}{k!}$

determines e.

Also:

$B = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

determines e.

Therefore:

$A \in U_e$

and

$B \in U_e$.

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4. Associated equivalence relation

For a fixed constant $\alpha \in R$, we define the relation:

$A \sim_\alpha B \Leftrightarrow \text{ valor }(A) = \text{ valor }(B) = \alpha$.

This relation identifies expressions by their common value, not by their syntactic form nor by their procedure of construction.

Theorem 1 — $\sim_\alpha$ is an equivalence relation

Let $\alpha \in R$ and let U_α be the set of constructions that determine α. Then $\sim_\alpha$ is an equivalence relation on U_α .

Proof

Reflexivity:

If $A \in U_\alpha$, then

$\text{ valor }(A) = \alpha$.

Therefore:

$A \sim_\alpha A$.

Symmetry:

If $A \sim_\alpha B$, then:

$\text{ valor }(A) = \text{ valor }(B) = \alpha$.

By symmetry of equality:

$\text{ valor }(B) = \text{ valor }(A) = \alpha$.

Thus:

$B \sim_\alpha A$.

Transitivity:

If $A \sim_\alpha B$ and $B \sim_\alpha C$, then:

$$\text{valor}\ (A) = \text{valor}\ (B) = \alpha$$

and

$$\text{valor}\ (B) = \text{valor}\ (C) = \alpha\ .$$

Therefore:

$$\text{valor}\ (A) = \text{valor}\ (C) = \alpha\ .$$

Thus:

$A \sim_\alpha C$.

Hence $\sim_\alpha$ is reflexive, symmetric and transitive. Consequently, it is an equivalence relation.

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5. Link between N and R

Let $\alpha \in R$.

Assume that there exists a sequence $(a_n)_{n \in N}$ such that:

$$\lim_{n \to \infty} a_n = \alpha\ .$$

Assume also that there exists a continuous process associated with a function F such that:

$$\lim_{dx \to 0} Q(F, x, dx) = \alpha\ .$$

Then both constructions belong to U_{α}.

The situation may be represented schematically as:

$$N \xrightarrow{n \to \infty} \alpha \xleftarrow{dx \to 0} R.$$

This notation does not mean:

$$N = R.$$

Nor does it mean:

$$N \cong R$$

as sets, topological spaces, fields or ordered structures.

It means only that two procedures defined from different natures can determine the same element α of R.

Theorem 2 — Discrete-continuous link

Let $\alpha \in R$. If there exists a discrete construction A_N indexed by N and a continuous construction A_R defined over R such that:

$$\text{valor}\,(A_N) = \alpha$$

and

$$\text{valor}\,(A_R) = \alpha,$$

then:

$$A_N \sim_{\alpha} A_R.$$

Consequently, A_N and A_R belong to the same class U_{α}.

Proof

By hypothesis:

$$\text{valor}\,(A_N) = \alpha$$

and

$$\text{valor}\,(A_R) = \alpha\,.$$

Then:

$$\text{valor}\,(A_N) = \text{valor}\,(A_R) = \alpha\,.$$

By definition of $\sim_\alpha$:

$$A_N \sim_\alpha A_R\,.$$

Therefore:

$$A_N, A_R \in U_\alpha\,.$$

This proves the claim.

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6. Application to *e*

The constant e is a central example because it appears naturally both in discrete constructions and in continuous characterizations.

6.1 Discrete construction

Consider the sequence:

$$a_n = (1 + \tfrac{1}{n})^n, \qquad n \in N\,.$$

Then:

$$\lim_{n \to \infty} (1 + \tfrac{1}{n})^n = e\,.$$

The construction is indexed by discrete values:

$$n = 1, 2, 3, \ldots$$

The value e appears as the limit when:

$n \to \infty$.

Therefore:

$N \xrightarrow{n \to \infty} e$.

This expression does not state that $e \in N$.

It states that a sequence indexed by N determines, in the limit, an element of R.

6.2 Construction by series

One also has:

$e = \sum_{k=0}^{\infty} \frac{1}{k!}$.

This representation also has a discrete structure, since it is indexed by $k \in N$.

The determined object is the same:

$\sum_{k=0}^{\infty} \frac{1}{k!} = e$.

6.3 Continuous characterization

Consider the function:

$f(x) = e^x$.

Its derivative is defined by:

$f'(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$.

Substituting:

$f'(x) = \lim_{dx \to 0} \frac{e^{x+dx} - e^x}{dx}$.

Factoring:

$$f'(x) = e^x \lim_{dx \to 0} \frac{e^{dx} - 1}{dx} .$$

The base e is characterized by:

$$\lim_{dx \to 0} \frac{e^{dx} - 1}{dx} = 1 .$$

Therefore:

$$\frac{d}{dx} e^x = e^x .$$

In particular:

$$e = e^1 .$$

Here e appears within a continuous structure, linked to the differential limit:

$$dx \to 0 .$$

Thus:

$$R \xrightarrow{dx \to 0} e .$$

6.4 Class U_e

We define:

$$U_e = \{E_i : \text{valor}(E_i) = e\} .$$

Examples of elements of U_e are:

$$\lim_{n \to \infty} (1 + \tfrac{1}{n})^n ,$$

$$\sum_{k=0}^{\infty} \frac{1}{k!},$$

exp(1),

$$\int_1^e \frac{1}{x} dx = 1$$

as an indirect characterization of e, that is, e is the unique positive real number satisfying:

$$\int_1^e \frac{1}{x}\, dx = 1 .$$

These expressions do not all have the same form. Some are limits, others are series, others are functional values and others are implicit characterizations.

However, all of them determine e.

Therefore, they belong to the same class U_e.

6.5 Known U_e and unknown U_e

The total set U_e can be conceptually divided into two regions:

$$U_e = U_e^{\text{known}} \cup U_e^{\text{unknown}} .$$

Here U_e^{known} represents the expressions already identified, verified or accepted as determinations of e within a given mathematical framework.

For example:

$$\exp(1) \in U_e^{\text{known}} .$$

Also:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \in U_e^{\text{known}} .$$

And:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \in U_e^{\text{known}} .$$

By contrast, U_e^{unknown} denotes the expressions not yet discovered, not classified or not proved to be equivalent to e.

This division is not ontological but epistemological.

The value e does not change because an expression is known or unknown.

What changes is our catalogue of recognized expressions.

6.6 Enumeration of expressions

If a formal language with a finite or countable alphabet and effective syntactic rules is fixed, the set of finite expressions of that language is countable.

Therefore, they can be syntactically enumerated:

$$E_1, E_2, E_3, \ldots$$

However, not every syntactic expression determines a real value.

And there is not always a general procedure that decides, for any sufficiently rich expression, whether:

$$\text{valor}(E_i) = e.$$

Therefore, there may exist a syntactic enumeration of candidates without a complete and effective classification of U_e.

This observation separates three things:

1. enumerating possible expressions;
2. evaluating well-defined expressions;
3. proving membership in U_e.

Corollary 1 — The known catalogue is partial

The set U_e^{known} does not necessarily exhaust U_e.

Proof

By definition, U_e^{known} contains the expressions whose membership in U_e has been recognized within a given framework.

But the definition of U_e does not require its elements to be known to us. It only requires that they determine e.

Therefore, unless a complete characterization of the language and of all its values is proved, one cannot conclude that:

$$U_e^{\text{known}} = U_e .$$

Hence the known catalogue must be treated as partial.

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7. Application to π

The constant π also appears as a link between discrete and continuous constructions.

7.1 Discrete series

A classical representation is:

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} .$$

Equivalently:

$$\pi = \lim_{n \to \infty} 4 \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} .$$

This construction is indexed by:

$$n \in N .$$

Thus:

$$N \xrightarrow{n \to \infty} \pi .$$

7.2 Continuous integral

One also has:

$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx$.

The integral is constructed by partitions of a real interval and the refinement of those partitions.

In terms of scale:

$\Delta x \to 0$.

Thus:

$R \xrightarrow{\Delta x \to 0} \pi$.

7.3 Class U_π

We define:

$U_\pi = \{P_i : \text{valor}\ (P_i) = \pi\}$.

Then the following belong to U_π:

$4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$,

$4\int_0^1 \frac{1}{1+x^2}\,dx$,

$\frac{C}{d}$

when C is the length of a Euclidean circumference and d its diameter.

These expressions are not identical as procedures, but they are equivalent with respect to the value they determine.

Therefore:

$N \xrightarrow{n \to \infty} \pi \xleftarrow{\Delta x \to 0} R$.

Corollary 2 — π as invariant of representations

The constant π acts as an invariant between geometric, analytic, integral and serial representations.

Proof

All the previous expressions determine the same value π under their respective hypotheses.

By definition of U_π:

$P_i \in U_\pi \Leftrightarrow \text{valor}(P_i) = \pi$.

Therefore, all of them belong to the same equivalence class.

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8. Application to φ

Let φ be the golden ratio:

$\varphi = \frac{1+\sqrt{5}}{2}$.

This constant admits an exact algebraic definition:

$\varphi^2 = \varphi + 1$.

It also appears as the limit of quotients of Fibonacci numbers.

Let (F_n) be the sequence defined by:

$F_0 = 0,$

$F_1 = 1,$

and

$F_{n+1} = F_n + F_{n-1}$.

Then:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi .$$

The construction by Fibonacci is discrete, since it depends on $n \in N$.

The algebraic construction belongs to the real field.

We define:

$$U_{\varphi} = \{\Phi_i : \text{valor } (\Phi_i) = \varphi\} .$$

Then:

$$\frac{1+\sqrt{5}}{2} \in U_{\varphi}$$

and

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} \in U_{\varphi} .$$

Again:

$$N \xrightarrow{n \to \infty} \varphi \xleftarrow{\text{algebra}} R .$$

The previous notation is schematic: it does not state that real algebra is a differential process. It only indicates that φ can be determined both from a discrete recurrence and from an exact equation in R.

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9. On $U_{\tilde{n}}$

In EN_49, the need appears to represent the conceptual passage between a natural index n and an extended, interpolated or represented form in a continuous environment. For this purpose, the notation $\tilde{n}$ is introduced.

The rigorous interpretation must be careful.

One must not state that a natural number n is identical to a continuous real.

One may consider an inclusion map:

$\iota : N \to R$

given by:

$\iota(n) = n$.

This inclusion allows each natural number to be seen as a particular real number, but it does not turn N into R.

One may also consider an interpolation or extension of a discrete sequence.

If one has a sequence:

$a : N \to R$,

one may seek a function:

$\tilde{a} : R_{\geq 0} \to R$

such that:

$\tilde{a}(n) = a_n$

for every $n \in N$.

In this context, $\tilde{n}$ does not represent an equality between the discrete and the continuous. It represents a controlled extension, coding or interpolation.

We therefore define:

$U_{\tilde{n}} = \{N_i : \text{valor } (N_i) = \tilde{n}\}$,

provided that $\tilde{n}$ has previously been defined within the system.

Observation

$U_{\tilde{n}}$ must not be used as proof that N and R are equivalent.

It must be used as a family of representations of a concrete extension or image of a discrete index within a broader construction.

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10. On U_g

Let g be a real magnitude determined by a model.

For example, if g represents local gravitational acceleration, then it must not simply be treated as an absolute universal constant, because its value depends on location, height and the physical model used.

In an idealized local framework one may fix:

$$g \approx 9.80665\,\mathrm{m}/\mathrm{s}^2 .$$

If the framework fixes an exact or conventional value of g, then one may define:

$$U_g = \{G_i : \text{valor}\,(G_i) = g\} .$$

A possible dynamical expression is:

$$s(t) = s_0 + v_0 t - \tfrac{1}{2} g t^2 .$$

From it one obtains:

$$\frac{d^2 s}{dt^2} = -g .$$

It may also appear through an idealized law of gravitation:

$$g = \frac{GM}{r^2},$$

under the hypotheses of central mass M, distance r and gravitational constant G.

Both expressions belong to U_g only if they determine the same value of g within the same physical framework and under the same conditions.

Rigorous warning

It is not correct to state without conditions that all expressions containing the symbol g belong to the same class.

Only those expressions that determine exactly the same value fixed by the model belong to U_g.

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11. Differentiation and integration as limit processes

Differentiation and integration appear in EN_49 as two complementary ways of operating with limits.

11.1 Differentiation

The derivative is defined by:

$f'(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$.

This process takes a variation:

dx

and studies its behavior when:

$dx \to 0$.

The derivative expresses an instantaneous rate of change.

11.2 Integration

The definite integral may be constructed as the limit of Riemann sums:

$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$.

Here a finite sum indexed by:

$$k = 1, 2, \dots, n\,.$$

appears.

The passage to the continuum is carried out through the refinement:

$$n \to \infty$$

and

$$\max_k \Delta x_k \to 0\,.$$

Thus, the integral joins a discrete structure of sums with a continuous object of area or accumulation.

11.3 Fundamental theorem of calculus

If f is continuous on $[a, b]$ and one defines:

$$F(x) = \int_a^x f(t)\,dt,$$

then:

$$F'(x) = f(x)$$

for every $x \in (a, b)$.

Moreover, if F is an antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

This result connects differentiation and integration.

Differentiation studies local variation.

Integration reconstructs global accumulation.

In EN_49, this relation reinforces the idea that apparently opposite processes can be compatible within the same structure.

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12. Exact classes and limit classes

To avoid ambiguity, it is useful to distinguish two subfamilies:

$$U_\alpha^{\text{exact}} = \{A\colon \text{valor}\,(A) = \alpha\}$$

and

$$U_\alpha^{\text{limit}} = \{(A_n)\colon \lim_{n\to\infty} A_n = \alpha\}\,.$$

This notation requires care.

A complete limit process may be an exact expression of α.

But the finite terms of the process are not, in general, exact expressions of α .

Thus:

$$\lim_{n\to\infty} A_n \in U_\alpha^{\text{exact}}$$

if the limit exists and equals α.

But:

$$A_n \notin U_\alpha^{\text{exact}}$$

for finite n, unless for some particular reason $A_n = \alpha$.

Therefore, one may write more finely:

$$U_\alpha = U_\alpha^{\text{direct}} \cup U_\alpha^{\text{limit}} \cup U_\alpha^{\text{implicit}} \cup U_\alpha^{\text{functional}},$$

where the categories may overlap.

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13. Generalization to U_{f_i}

So far constants have been considered. But the idea can be generalized to functions.

Let $f_i : D \to C$ be a function defined on a domain D with codomain C.

We define:

$$U_{f_i} = \{F_j : \text{valor}\,(F_j) = f_i\},$$

where $\text{valor}\,(F_j) = f_i$ means that F_j determines the same function as f_i on the relevant domain.

More explicitly:

$$F_j \in U_{f_i} \Leftrightarrow \forall x \in D,\ F_j(x) = f_i(x).$$

If one works with functions defined almost everywhere, distributions, equivalence classes of functions or restricted domains, the corresponding equality criterion must be declared.

Example

Let:

$$f(x) = e^x.$$

Then the following belong to U_f:

$$x \mapsto e^x,$$

$$x \mapsto \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and the solution of the differential problem:

$$y' = y,$$

$$y(0) = 1.$$

These three representations determine the same function on R.

Therefore, they belong to the same class U_f.

Theorem 3 — Functional equivalence

Let D be a domain and let $F, G{:}D \to C$. If:

$\forall x \in D,\ F(x) = G(x),$

then F and G belong to the same functional equivalence class.

Proof

By hypothesis, F and G have the same value at each point of D.

Therefore, they determine the same function as a relation between elements of D and elements of C.

Thus:

$F \sim_f G$.

This proves the claim.

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14. Main theorems and corollaries

Theorem 4 — Value invariant

Let $\alpha \in R$. If A and B are well-defined mathematical constructions and:

$\text{valor}(A) = \text{valor}(B) = \alpha,$

then A and B are equivalent with respect to U_α.

Proof

This is immediate by the definition of $\sim_\alpha$.

Corollary 3 — Distinction of procedure

Two expressions may belong to the same class U_α without being the same procedure.

Proof

Membership in U_α requires equality of value, not syntactic equality nor equality of method.

Therefore, if two different procedures determine α, both belong to U_α even though they remain different procedures.

Corollary 4 — No structural equivalence between N and R

The existence of a discrete-continuous link for a constant α does not imply that N and R are equivalent as structures.

Proof

From the fact that:

$$\lim_{n \to \infty} a_n = \alpha$$

and

$$\lim_{dx \to 0} Q(F, x, dx) = \alpha$$

it follows only that both constructions determine the same value α.

No structural bijection, homeomorphism, field isomorphism, equivalence of complete order or equality between N and R follows from this.

Therefore, the valid conclusion is common membership in U_α, not identity between domains.

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15. Conjectures and extensions

The following ideas must be treated as conjectures or research programs, not as theorems proved in this document.

Conjecture 1 — Open growth of U_α^{known}

For mathematically rich constants such as e, π and φ, the set of known expressions that determine them may grow indefinitely with new frameworks, new identities and new translations between theories.

This does not state that U_α is literally inexhaustible in every fixed formal language.

It states that, in mathematical practice, the recognized catalogue of expressions may be expanded without the value α changing.

Conjecture 2 — Compatibility nodes

Constants such as e, π and φ may be interpreted as compatibility nodes between different regions of mathematics.

For example, π connects geometry, analysis, trigonometry, series and integration.

e connects growth, differential calculus, series, discrete limits and differential equations.

φ connects algebra, recurrences, proportions and limits of sequences.

This reading is structural and conceptual. It does not replace the particular proofs of each identity.

Conjecture 3 — UNIHOLOG extension

In the UNIHOLOG framework, each family U_{f_i} may be studied as a network of equivalent expressions that reveals connections between discrete, continuous, algebraic, geometric, dynamical and physical domains.

The demonstrable part consists in verifying, case by case, that two expressions determine the same object.

The conjectural part consists in interpreting the complete network as an architecture of compatibility between theories.

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16. Final synthesis

EN_49 does not state that N and R are the same.

It states something subtler and mathematically defensible:

$$N \xrightarrow{n \to \infty} \alpha \xleftarrow{dx \to 0} R$$

when there exist discrete and continuous constructions that determine the same $\alpha \in R$.

The constant α acts as an invariant.

The class U_α gathers the expressions that determine that same value.

In the case of e:

$$N \xrightarrow{n \to \infty} e \xleftarrow{dx \to 0} R.$$

In the case of π:

$$N \xrightarrow{n \to \infty} \pi \xleftarrow{\Delta x \to 0} R.$$

In the case of φ:

$$N \xrightarrow{n \to \infty} \varphi \xleftarrow{\text{algebra}} R.$$

In the general case:

$$U_{f_i} = \{F_j : \forall x \in D,\ F_j(x) = f_i(x)\}.$$

The power of EN_49 lies in carefully distinguishing:

1. domain of construction;
2. procedure;
3. expression;
4. limit process;
5. exact value;
6. equivalence class;
7. conjectural interpretation.

With this separation, the bridge between discrete and continuous ceases to be a vague claim and becomes a rigorous structure:

different paths can preserve the same object.

That common object is the link.